Resit Exam Calculus 2

2014

university of groningen

The exam consists of 4 problems. You have 180 minutes to answer the questions. You can achieve 100 points which includes a bonus of 10 points.

1. [**10 Points**] Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function, and let
$$w(x, y, z) = f\left(\frac{y - x}{xy}, \frac{z - x}{xz}\right).$$
Show that
$$x^2 \frac{\partial w}{\partial x} + y^2 \frac{\partial w}{\partial y} + z^2 \frac{\partial w}{\partial z} = 0.$$

2. [**10+5 Points**] Let C be the curve parametrized by $\mathbf{r} : [0, 2\pi] \to \mathbb{R}^3$,
$$\mathbf{r}(t) = (\sin t - t\cos t)\,\mathbf{i} + (\cos t + t\sin t)\,\mathbf{j} + 2\,\mathbf{k}.$$
   (a) Find the parametrization of C by arc length.
   (b) For each point on C, compute the curvature of C at this point.

3. [**7+8 Points**] Consider the ellipsoid
$$x^2 + 2y^2 + 3z^2 = 6.$$
   (a) Compute the tangent plane of the ellipsoid at the point $(x, y, z) = (-1, -1, -1)$.
   (b) Use the Method of Lagrange Multipliers to find the radius of the smallest sphere centered at the origin for which the ellipsoid fits inside.

4. [**12+8 Points**] Let a, b and c be functions from $\mathbb{R}$ to $\mathbb{R}$ of class C^1.
   (a) Show that
$$\mathbf{F} = (a(x) + y + z)\,\mathbf{i} + (x + b(y) + z)\,\mathbf{j} + (x + y + c(z))\,\mathbf{k}$$
   is conservative, and determine a potential function for $\mathbf{F}$.
   (b) For $a(x) = x$, $b(y) = y^2$ and $c(z) = z^3$, compute the line integral along the straight line segment starting at the point $\mathbf{p} = \mathbf{i} + \mathbf{j}$ and ending at the point $\mathbf{q} = \mathbf{j} + \mathbf{k}$. Verify this result using the potential function found in part (a) .

5. [**6+4 Points**]
   (a) Give the precise statements of Stokes' and Green's Theorems.
   (b) Show that Green's Theorem follows from Stokes' Theorem.

6. [**20 Points**] Let S be the surface defined by $z = \mathrm{e}^{1-x^2-y^2}$, $z \geq 1$, oriented by the upward normal, and let
$$\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + (2 - 2z)\,\mathbf{k}.$$
Use Gauß' Theorem to calculate the flux of $\mathbf{F}$ through S.

Solutions

1. Let

$$u = \frac{y-x}{xy} \text{ and } v = \frac{z-x}{xz}.$$

Then, by the chain rule,

$$\begin{aligned}
\frac{\partial w}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}, \\
\frac{\partial w}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}, \\
\frac{\partial w}{\partial z} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z}.
\end{aligned}$$

Straightforward computations give

$$\frac{\partial u}{\partial x} = -\frac{1}{x^2}, \quad \frac{\partial u}{\partial y} = \frac{1}{y^2}, \quad \frac{\partial u}{\partial z} = 0, \quad \frac{\partial v}{\partial x} = -\frac{1}{x^2}, \quad \frac{\partial v}{\partial y} = 0, \quad \frac{\partial v}{\partial z} = \frac{1}{z^2}.$$

Thus

$$\begin{aligned}
&x^2\frac{\partial w}{\partial x} + y^2\frac{\partial w}{\partial y} + z^2\frac{\partial w}{\partial z} \\
=&x^2\big(\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}\big) + y^2\big(\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}\big) + z^2\big(\frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z}\big) \\
=&x^2\big(\frac{\partial f}{\partial u}(-\frac{1}{x^2}) + \frac{\partial f}{\partial v}(-\frac{1}{x^2})\big) + y^2\big(\frac{\partial f}{\partial u}\frac{1}{y^2} + \frac{\partial f}{\partial v}0\big) + z^2\big(\frac{\partial f}{\partial u}0 + \frac{\partial f}{\partial v}\frac{1}{z^2}\big) \\
=&\frac{\partial f}{\partial u}(-1+1+0) + \frac{\partial f}{\partial v}(-1+0+1) \\
=&0.
\end{aligned}$$

2. (a) The arc length is defined as

$$s(t) = \int_0^t |\mathbf{r}'(\tau)|\,\mathrm{d}\tau\,.$$

We have $\mathbf{r}'(t) = (\cos t - \cos t + t\sin t)\,\mathbf{i} + (-\sin t + \sin t + t\cos t)\,\mathbf{j} + 0\,\mathbf{k} = t\sin t\,\mathbf{i} + t\cos t\,\mathbf{j}$, and hence

$$|\mathbf{r}'(t)| = \sqrt{t^2\sin^2 t + t^2\cos^2 t} = |t| = t \text{ for } t \geq 0.$$

Hence

$$s(t) = \int_0^t \tau\,\mathrm{d}\tau = \frac{1}{2}t^2.$$

Solving for t gives

$$t(s) = \sqrt{2s}.$$

So the parametrization of C by arc length is given by

$$\begin{aligned}
\tilde{\mathbf{r}}(s) = \mathbf{r}(t(s)) = \mathbf{r}(t) &= (\sin t(s) - t(s)\cos t(s))\,\mathbf{i} + (\cos t(s) + t(s)\sin t(s))\,\mathbf{j} + 2\,\mathbf{k} \\
&= (\sin\sqrt{2s} - \sqrt{2s}\cos\sqrt{2s})\,\mathbf{i} + (\cos\sqrt{2s} + \sqrt{2s}\sin\sqrt{2s})\,\mathbf{j} + 2\,\mathbf{k}
\end{aligned}$$

where $0 \leq s \leq 2\pi^2 = s(2\pi)$.

(b) The curvature κ is defined as

$$\kappa = \left| \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} \right|,$$

where $\mathbf{T}$ is the unit tangent vector. By the chain rule

$$\kappa = \frac{1}{|\mathbf{r}'(t)|} \left| \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t} \right|,$$

From part (a) we get

$$\mathbf{T} = \frac{1}{|\mathbf{r}'(t)|}\mathbf{r}'(t) = \frac{1}{t}\big(t \sin t\,\mathbf{i} + t\cos t\,\mathbf{j}\big) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

which gives

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t} = \cos t\,\mathbf{i} - \sin t\,\mathbf{j}$$

and hence

$$\left| \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t} \right| = 1.$$

The curvature of C at $\mathbf{r}(t)$ is thus

$$\kappa = \frac{1}{t}.$$

3. (a) *First method:* The 'lower' half of the ellipsoid can be considered to be the graph of the function

$$f(x, y) = -\sqrt{6 - (3x^2 + 2y^2)}.$$

We can thus compute tangent plane of the ellipsoid at $(x, y, z) = (-1, -1, -1)$ from the linearization of f at $(x, y) = (-1, -1)$ which is given by

$$L(x, y) = f(-1, -1) + f_x(-1, -1)(x + 1) + f_y(-1, -1)(y + 1).$$

Using $f_x(x, y) = 3x/\sqrt{6 - 3x^2 - 2y^2}$ and $f_y(x, y) = 2y/\sqrt{6 - 3x^2 - 2y^2}$ and hence $f_x(-1, -1) = -3$ and $f_y(-1, -1) = -2$ we find for the tangent plane

$$z = L(x, y) = -1 + 3(x + 1) + 2(y + 1) = 4 + 3x + 2y.$$

Second method: We can view the ellipsoid to be given by a level set of the function

$$F(x, y, z) = 3x^2 + 2y^2 + z^2,$$

and we can hence write the tangent plane as

$$\nabla F(-1, -1, -1) \cdot (x + 1, y + 1, z + 1) = 0.$$

We have $\nabla F(x, y, z) = (6x, 4y, 2z)$ and hence $\nabla F(-1, -1, -1) = (-6, -4, -2)$. For the tangent plane we thus find

$$(-6, -4, -2) \cdot (x + 1, y + 1, z + 1) = 0,$$

or equivalently,

$$z = 4 + 3x + 2y$$

which agrees with the result obtained from the first method.

(b) Let

$$g(x,y,z) = x^2 + y^2 + z^2$$

which gives the square distance of (x,y,z) to the origin. To find the smallest sphere for which the ellipsoid fits inside we study the extrema of g under the constraint $F(x,y,z) = 3x^2 + 2y^2 + z^2 = 6$. By the method of Langrange multipliers there is a $\lambda \in \mathbb{R}$ such that $\nabla g = \lambda \nabla F$ at the extremum. This yields the set of equations

$$\begin{aligned} g_x &= \lambda F_x, \\ g_y &= \lambda F_y, \\ g_z &= \lambda F_z, \\ F(x,y,z) &= 6, \end{aligned}$$

i.e.

$$\begin{aligned} 2x &= \lambda 6x, \\ 2y &= \lambda 4y, \\ 2z &= \lambda 2z, \\ 3x^2 + 2y^2 + z^2 &= 6, \end{aligned}$$

which is equivalent to

$$\begin{aligned} x = 0 \cup \lambda &= \tfrac{1}{3}, \\ y = 0 \cup \lambda &= \tfrac{1}{2}, \\ z = 0 \cup \lambda &= 1, \\ 3x^2 + 2y^2 + z^2 &= 6. \end{aligned}$$

This in turn is equivalent to

$$\begin{gathered} x = y = 0 \cap \lambda = 1 \text{ or } x = z = 0 \cap \lambda = \tfrac{1}{2} \text{ or } y = z = 0 \cap \lambda = \tfrac{1}{3}, \\ 3x^2 + 2y^2 + z^2 = 6 \end{gathered}$$

or

$$\begin{aligned} x = y = 0, \quad & z = \pm\sqrt{6}, \quad \lambda = 1 \text{ or} \\ x = z = 0, \quad & y = \pm\sqrt{3}, \quad \lambda = \tfrac{1}{2} \text{ or} \\ y = z = 0, \quad & x = \pm\sqrt{2}, \quad \lambda = \tfrac{1}{3}. \end{aligned}$$

We have $g(0,0,\pm\sqrt{6}) = 6$, $g(0,\pm\sqrt{3},0) = 3$ and $g(\pm\sqrt{2},0) = 2$. The smallest sphere for which the ellipsoid fits inside has radius where g has a maximium on the ellipsoid, i.e. the sphere has radius $\sqrt{6}$ which agrees with the largest semi axis of the ellipsoid.

4. (a) Since $\mathbf{F}$ is defined on a simply connected domain it is sufficient to show that $\nabla \times \mathbf{F} = 0$ in order to prove that $\mathbf{F}$ is conservative.:

$$\nabla \times \mathbf{F}(x,y,z) = (\partial_y F_z - \partial_z F_y, \partial_z F_x - \partial_x F_z, \partial_x F_y - \partial_y F_x) = (1-1, 1-1, 1-1) = 0.$$

The potential function $f : \mathbb{R}^3 \to \mathbb{R}$ satisfies $\nabla f(x,y,z) = \mathbf{F}$, i.e.

$$\frac{\partial f}{\partial x} = a(x) + y + z\,, \tag{1}$$

$$\frac{\partial f}{\partial y} = x + b(y) + z\,, \tag{2}$$

$$\frac{\partial f}{\partial z} = x + y + c(z)\,. \tag{3}$$

From Eq. (1) we get $\frac{\partial f}{\partial x} = a(x) + y + z$. Integrating with respect to x gives $f(x, y, z) = A(x) + yx + zx + g(y, z)$, where A is an integral of a (note that different choices for A differ only by constants which can be absorbed in the function g). Differentiating this f with respect to y should agree with the right hand side of Eq. (2). Equating the two gives $\frac{\partial g(y,z)}{\partial y} = b(y) + z$. Integrating with respect to y gives $g(y, z) = B(y)+zy+h(z)$ where B is an integral of b (similarly to above different choices for B differ by constants which can be absorbed in the function h). Hence $f(x, y, z) = A(x)+yx+zx+B(y)+yz+h(z)$. Differentiating this f with respect to z should agree with the right hand side of Eq. (3). Equating the two gives $c(z) = h'(z)$. Integrating with respect to z gives $h(z) = C(z) + d$ where $d \in \mathbb{R}$ is a constant. Hence

$$f(x, y, z) = A(x) + B(y) + C(z) + xy + xz + yz + d\,.$$

(b) Here $\mathbf{F}(x, y, z) = (x + y + z, x + y^2 + z, x + y + z^3)$. As a parametrization of the straight line segment connecting $\mathbf{p}$ and $\mathbf{q}$ we choose $\mathbf{r} : [0, 1] \to \mathbb{R}^3$, $t \mapsto (1 - t)\mathbf{p} + t\mathbf{q} = (1 - t, 1, t)$. Hence

$$\begin{aligned}\int_{\mathbf{pq}} \mathbf{F} \cdot \mathrm{d}\mathbf{s} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t = \int_0^1 (2, 2, 2 - t + t^3) \cdot (-1, 0, 1)\,\mathrm{d}t \\ &= \int_0^1 (t^3 - t)\,\mathrm{d}t = [\frac{1}{4}t^4 - \frac{1}{2}t^2]_0^1 = -\frac{1}{4}\,.\end{aligned}$$

Following part (a) the potential function is $f(x, y, z) = \frac{1}{2}x^2 + \frac{1}{3}y^3 + \frac{1}{4}z^4 + xy + xz + yz + d$. Hence $f(q) - f(p) = \frac{1}{3} + \frac{1}{4} - \left(\frac{1}{2} + \frac{1}{3}\right) = -\frac{1}{4}$.

5. See Stewart's book.

6. Gauß' Theorem relates a triple integral over a simple solid region to a surface integral over its boundary. In our case, we first need to 'close up' the surface S before it can be the boundary of a simple solid region. At height $z = 1$, the surface $z = \mathrm{e}^{1-x^2-y^2}, z \geq 1$ is the circle $x^2+y^2 = 1$. Let S' be the closed disk given by $x^2+y^2 \leq 1$, $z = 1$ oriented by downward-pointing normal (i.e. $-\mathbf{k}$). Then S' and S enclose a simple solid region we call E. Then the boundary of E is given by $S' \cup S$ and E induces the same orientation on S and S' that they already had. Gauß' Theorem now yields

$$\iiint_E \nabla \cdot \mathbf{F}\,\mathrm{d}V = \iint_S \mathbf{F} \cdot\, \mathrm{d}\mathbf{S} + \iint_{S'} \mathbf{F} \cdot\, \mathrm{d}\mathbf{S}.$$

We have $\mathbf{F} = x\,\mathbf{i}+y\,\mathbf{j}+(2-2z)\,\mathbf{k}$, so that $\nabla\cdot\mathbf{F} = 1+1-2 = 0$. Hence $\iiint_E \nabla\cdot\mathbf{F}\,\mathrm{d}V = 0$, so that

$$\iint_S \mathbf{F} \cdot\, \mathrm{d}\mathbf{S} = -\iint_{S'} \mathbf{F} \cdot\, \mathrm{d}\mathbf{S}. = -\iint_{S'} \mathbf{F} \cdot -\mathbf{k}\,\mathrm{d}S = \iint_{S'} \mathbf{F} \cdot \mathbf{k}\,\mathrm{d}S = \iint_{S'} 2-2z\,\mathrm{d}S = 0,$$

because we have $2 - 2z = 0$ on S'.